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UNIFORM HYPERBOLICITY OF THE CURVE GRAPHS

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ABSTRACT. We show that there is a universal constant, k , such that the curve graph associated to any compact orientable surface is k -hyperbolic. Independent proofs of this have been given by Aougab, by Hensel, Przytycki and Webb, and by Clay, Rafi and Schleimer.

1. INTRODUCTION

Let Σ be a closed orientable surface of genus g , together with a (possibly empty) finite set $\Pi \subseteq \Sigma$. Set $p = |\Pi|$. We assume that $3g + p \geq 5$. Let $\mathcal{G} = \mathcal{G}(g, p)$ be the curve graph associated to (Σ, Π) ; that is, the 1-skeleton of the curve complex as originally defined in [Ha]. Its vertex set, $V(\mathcal{G})$, is the set of free homotopy classes of non-trivial non-peripheral closed curves in $\Sigma \setminus \Pi$; and two such curves are deemed to be adjacent in \mathcal{G} if they can be realised disjointly in $\Sigma \setminus \Pi$. These, and related, complexes are now central tools in geometric group theory and hyperbolic geometry.

In [MM1], it was shown that, for all g, p , $\mathcal{G}(g, p)$ is hyperbolic in the sense of Gromov [Gr]. In [B], it was shown that the hyperbolicity constant, k , is bounded above by a function that is logarithmic in $g + p$. In fact, we show here that k can be chosen independently of g and p :

Theorem 1.1. *There is a universal constant, $k \in \mathbb{N}$, such that $\mathcal{G}(g, p)$ is k -hyperbolic for all g, p with $3g + p \geq 5$.*

We will give some estimates for k (though certainly not optimal) in Section 4.

Independent proofs of this result have been found by Aougab [A], by Hensel, Przytycki and Webb [HePW], and by Clay, Rafi and Schleimer [CRS]. The proofs in [HePW] and [CRS] are both combinatorial in nature. The proof in [A] is based on broadly similar principles to those described here, though the specifics are different. Both this paper and [A] make use of riemannian geometry. The argument of [HePW] seems to give the optimal constants.

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Given Theorem 1.1, one can also obtain uniform bounds for the Bounded Geodesic Image Theorem of [MM2]. For this, one can combine the description of quasigeodesic lines in [B] with an unpublished argument of Leininger. In fact, a more direct approach, just using hyperbolicity, has recently been found by Webb [Web].

We remark that Theorem 1.1 does not imply uniform hyperbolicity of the curve complexes (with simplices realised as regular euclidean simplices) since their 1-skeleta are not uniformly quasi-isometrically embedded — there is an arbitrarily large contraction of distances as the complexity increases.

The proof of Theorem 1.1 consists primarily of going through the arguments of [B] with more careful bookkeeping of constants. This is accomplished in Section 2 here. In Sections 3 and 4 here, we show that much of this can be bypassed. In fact, we only really need a few results from [B], notably Lemmas 1.3, 4.4 and 4.5, together with the construction of singular euclidean structures described in Section 5 thereof.

We were motivated to look again at that paper after reading some estimates in [T] which relate distances to intersection number.

2. PROOFS

In this section, we will prove Proposition 2.6, which, together with Proposition 3.1 of [B] implies Theorem 1.1.

We will use the following different measures of the “complexity” of Σ, Π , tailored to different parts of the argument: $\xi_0 = 2g + p - 4$, $\xi_1 = 2g + p - 1$, $\xi_2 = 2g + p + 6$. For $\alpha, \beta \in V(\mathcal{G})$, we write $\iota(\alpha, \beta)$ for the intersection number, and $d(\alpha, \beta)$ for the combinatorial distance in the curve graph.

Lemma 2.1. *If $\gamma, \delta \in V(\mathcal{G})$, with $\iota(\gamma, \delta) \leq \xi_0 + 1$, then $d(\gamma, \delta) \leq 2$.*

Proof. We realise γ, δ in $\Sigma \setminus \Pi$ so that $|\gamma \cap \delta| = \iota(\gamma, \delta) = n$, say. Now, $\gamma \cup \delta$ is a graph with n vertices and $2n$ edges, and hence Euler characteristic $-n$. If $d(\gamma, \delta) > 2$, then $\gamma \cup \delta$ fills $\Sigma \setminus \Pi$ and so this Euler characteristic must be at most that of $\Sigma \setminus \Pi$, namely, $2 - 2g - p$. Thus, $n \geq 2g + p - 2$. Taking the contrapositive, if $n \leq \xi_0 + 1 = 2g + p - 3$, then $d(\gamma, \delta) \leq 2$. \square

Now, Lemma 1.3 of [B] shows that if $\alpha, \beta \in V(\mathcal{G})$ with $2\iota(\alpha, \beta) \leq ab$ for $a, b \in \mathbb{N}$, then there is some $\gamma \in V(\mathcal{G})$ with $\iota(\alpha, \gamma) \leq a$ and $\iota(\beta, \gamma) \leq b$. Applying this q times, together with Lemma 2.1, we get:

Corollary 2.2. *If $q \in \mathbb{N}$ and $\alpha, \beta \in V(\mathcal{G})$ with $2^q \iota(\alpha, \beta) \leq \xi_0^{q+1}$, then $d(\alpha, \beta) \leq 2(q + 1)$.*

Definition. By a *region* in Σ , we mean a subsurface, $H \subseteq \Sigma$, with $\partial H \cap \Pi = \emptyset$. A region is *trivial* if it is a topological disc containing at most one point of Π . An *annulus* in Σ is a region $A \subseteq \Sigma \setminus \Pi$ homeomorphic to $S^1 \times [0, 1]$ such that no component of $\Sigma \setminus A$ is trivial.

The core curve of an annulus therefore determines an element of $V(\mathcal{G})$.

Suppose that ρ is a riemannian metric on Σ . We allow for a finite number of cone singularities (which need bear no relation to Π). We define the *width* of an annulus $A \subseteq \Sigma$ to be the length of a shortest path in A connecting its two boundary components.

The following lemma is a slight variation of Lemma 5.1 of [B]. We follow a similar argument, but taking more care with constants.

The proof will make use of the following notion. Let α be an essential non-peripheral closed curve in $\Sigma \setminus \Pi$.

Definition. A *bridge* (across α) is an arc, $\delta \subseteq \Sigma \setminus \Pi$, with $\partial\delta = \delta \cap \alpha$ such that no component of $\Sigma \setminus (\alpha \cup \delta)$ is a disc not meeting Π .

In other words, $\alpha \cup \delta$ is an embedded π_1 -injective theta-curve in $\Sigma \setminus \Pi$, i.e. it is the union of three arcs which meet precisely in their endpoints and are pairwise non-homotopic relative to their endpoints.

Lemma 2.3. *Suppose that ρ is a (singular) riemannian metric on Σ , with $\text{area}(\Sigma) = 1$. Suppose that $3g + p \geq 5$. Suppose that there is a constant $h > 0$ such that for any trivial region $\Delta \subseteq \Sigma$ we have $\text{area}(\Delta) \leq h(\text{length}(\partial\Delta))^2$. Then Σ contains an annulus of width at least $\eta = 1/4\xi_1\xi_2\sqrt{h}$.*

Proof. To avoid technical details obscuring the exposition, we will relax inequalities so that they are assumed to hold up to an arbitrarily small additive constant $\epsilon > 0$. Thus, for example, a “shortest” curve will be assumed to be shortest to within ϵ etc. This will allow us, for example, to adjust paths so that they can be assumed to avoid Π . Finally, we can allow $\epsilon \rightarrow 0$. In what follows any “curve” in $\Sigma \setminus \Pi$ will be assumed to be essential and non-peripheral, i.e. it does not bound a trivial region in Σ .

Let $\eta_0 = 1/4\xi_2\sqrt{h}$. We claim that there are curves, $\alpha, \beta \subseteq \Sigma \setminus \Pi$ with $\rho(\alpha, \beta) \geq \eta_0$. Given this, we let $\phi : \Sigma \rightarrow [0, \eta_0] = [0, \xi_1\eta]$ be a 1-lipschitz map with $\alpha \subseteq \phi^{-1}(0)$ and $\beta \subseteq \phi^{-1}(\xi_1\eta)$. Given any $i \in \{1, \dots, \xi_1 - 1\}$, we can find a multicurve, $\gamma_i \subseteq \phi^{-1}(i\eta)$, which separates Σ into exactly two components, S_i^α, S_i^β , containing α and β respectively. We can assume $\gamma_i \cap \Pi = \emptyset$, and that $S_i^\alpha \subseteq S_{i+1}^\alpha$ for all i . These multicurves cut Σ into ξ_1 regions $M_i = S_i^\alpha \cap S_{i-1}^\beta$ (where $M_0 =$

S_1^α and $M_{\xi_1} = S_{\xi_1-1}^\beta$). At least one of these must have a component which is an annulus (otherwise each $M_i \setminus \Pi$ would have negative Euler characteristic, giving the contradiction that the Euler characteristic of $\Sigma \setminus \Pi$ is at most $-\xi_1 < 2 - 2g - p$). This annulus must have width at least η as required.

To find α, β , we take α to be a shortest curve in $\Sigma \setminus \Pi$. We suppose, for contradiction, that if $\beta \subseteq \Sigma \setminus \Pi$ is any curve, then $\rho(\alpha, \beta) < \eta_0$. Let $\lambda = 2\eta_0$.

We first claim that there is a collection disjoint bridges, $\delta_1, \dots, \delta_n$, across α with $\text{length}(\delta_i) < \lambda$ for all i and with each component of $\Sigma \setminus (\alpha \cup \delta_1 \cup \dots \cup \delta_n)$ trivial. (An example is shown in Figure 1.)

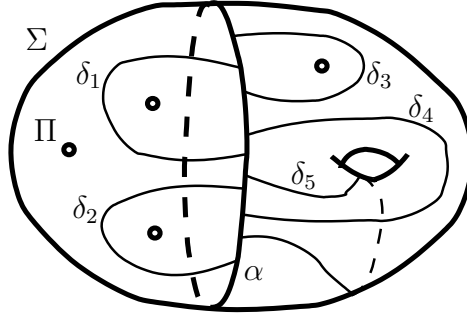


FIGURE 1. Example of a curve with bridges, $(g, p) = (1, 4)$

To prove this claim, let $N(\alpha, t)$ be the metric t -neighbourhood of α in Σ . Let $G(t)$ be the image of $\pi_1(N(\alpha, t) \setminus \Pi)$ in $\pi_1(\Sigma \setminus \Pi)$. Note that $G(0)$ is infinite cyclic, and $G(\eta_0) = \pi_1(\Sigma \setminus \Pi)$. As t increases from 0 to η_0 , $G(t)$ gets bigger at certain critical times, t_1, \dots, t_n . At these times, we can suppose we have added another generator, which we can represent as a bridge, δ_i , of length at most $2t_i < 2\eta_0 = \lambda$. Thus, inductively, $G(t_i)$ is supported on $\alpha \cup \delta_1 \cup \dots \cup \delta_i$. It follows that $\alpha \cup \delta_1 \cup \dots \cup \delta_n$ must fill $\Sigma \setminus \Pi$ (that is, carries all of $\pi_1(\Sigma \setminus \Pi)$), otherwise we could find a curve, β , with $\rho(\alpha, \beta) \geq \eta_0$. This gives us our collection of bridges as claimed.

Let $l = \text{length}(\alpha)$. We now claim that $l \leq 6\lambda$. So, suppose, to the contrary, that $l > 6\lambda$.

Given any i , write $\alpha = \alpha_i \cup \alpha'_i$, where α_i and α'_i are respectively the shorter and longer arcs with endpoints at $\partial\delta_i$. Thus, $\text{length}(\alpha_i) \leq l/2$, so $\text{length}(\alpha_i \cup \delta_i) \leq (l/2) + \lambda < l$, and so, by minimality of α , $\alpha_i \cup \delta_i$ must be trivial or peripheral, i.e. it bounds a trivial region in Σ . This

region must be a disc containing exactly one point of Π . Since this is true of all bridges δ_i , we already get a contradiction if $g > 0$ (and we can deduce that $l \leq 3\lambda$ in this case). So can assume that $g = 0$, and so α cuts Σ into two discs, H_0, H_1 . We have $|\Pi \cap H_i| \geq 2$, and we can assume that $|\Pi \cap H_0| \geq 3$.

Note also, if $\alpha'_i \cup \delta_i$ is non-trivial, then $\text{length}(\alpha'_i \cup \delta_i) \geq \text{length}(\alpha)$ and so $\text{length}(\alpha_i) \leq \text{length}(\delta_i) < \lambda$.

Now H_0 must contain at least two bridges from our collection. We can assume these are δ_1 and δ_2 . Recall that $\delta_1 \cap \delta_2 = \emptyset$. From the above, it follows that $\text{length}(\alpha_1) < \lambda$ and $\text{length}(\alpha_2) < \lambda$. Since δ_1 and δ_2 cannot cross, we must have $\alpha_1 \cap \alpha_2 = \emptyset$.

Now let δ_3 be a bridge in H_1 . As before, $\text{length}(\alpha_3) \leq l/2$, and so $\text{length}(\alpha_i \cup \alpha_3 \cup \delta_i \cup \delta_3) \leq 3\lambda + (l/2)$ for $i = 1, 2$. Now $\alpha_1 \cap \alpha_3 = \emptyset$ (otherwise, $\alpha_1 \cup \alpha_3 \cup \delta_1 \cup \delta_3$ would contain a curve of length at most $3\lambda + (l/2) < l$). Similarly, $\alpha_2 \cap \alpha_3 = \emptyset$. Now, given $i, j \in \{1, 2, 3\}$, let α_{ij} be the component of $\alpha \setminus (\alpha_1 \cup \alpha_2 \cup \alpha_3)$ between α_i and α_j . (See Figure 2.) Let θ_{ij} be the curve in Σ with image $\alpha_{ij} \cup \alpha_i \cup \alpha_j \cup \delta_i \cup \delta_j$, which passes through α_{ij} exactly twice. Together, the curves θ_{12}, θ_{23} and θ_{31} pass twice through each edge of $\alpha \cup \delta_1 \cup \delta_2 \cup \delta_3$, and so their lengths sum to at most $2l + 6\lambda$. We arrive at the contradiction that the length of at least one of the θ_{ij} is at most $\frac{1}{3}(2l + 6\lambda) < l$.

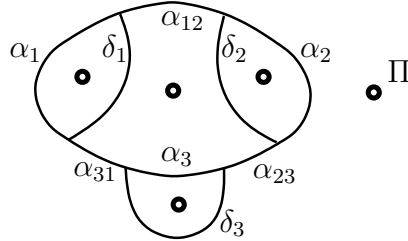


FIGURE 2. Picture of three bridges, $(g, p) = (0, 5)$

This shows that $l \leq 6\lambda$ as claimed.

After removing some of the bridges if necessary, we can assume that at most two of the complementary components are discs not meeting Π , and so $n \leq 2g + p$. Let $\sigma = \alpha \cup \delta_1 \cup \dots \cup \delta_n$. Thus $\text{length}(\sigma) < 6\lambda + n\lambda = (n + 6)\lambda \leq (2g + p + 6)\lambda = \xi_2\lambda$.

Since each component of $\Sigma \setminus \sigma$ is trivial, we must have $\text{area}(\Sigma) \leq h(2\text{length}(\sigma))^2$ (the worst case being when $\Sigma \setminus \sigma$ is connected). But we have assumed that $\text{area}(\Sigma) = 1$ and so $1 < h(2\xi_2\lambda)^2$. Now, $\lambda = 2\eta_0 = 2(1/4\xi_2\sqrt{h}) = 1/2\xi_2\sqrt{h}$, so we arrive at the contradiction that $1 < 1$.

This shows that there must be a curve, β , in $\Sigma \setminus \Pi$ with $\rho(\alpha, \beta) \geq \eta_0$ as claimed. \square

In fact, the argument also applies if $(g, p) = (1, 1)$. If $(g, p) = (0, 4)$, we will only need to consider a special case, namely, the quotient of a euclidean torus by an involution with four fixed points. In that case, we can set $\eta = 1/2$.

We will now set $h = 1/2\pi$. This gives $\eta = 1/4\xi_1\xi_2\sqrt{1/2\pi} = \sqrt{2\pi}/4\xi_1\xi_2$. As in Section 5 of [B], we define $R = \sqrt{2}/\eta$. In this case therefore, $R = (4/\sqrt{\pi})\xi_1\xi_2$.

Now suppose that α, β are weighted multicurves in the sense defined in [B]. (In other words, each is a measured lamination whose support is a disjoint union of curves.)

Definition. The *weighted intersection number*, $\iota(\alpha, \beta)$, of α and β is the sum $\sum_{i,j} \lambda_i \lambda_j \iota(\alpha_i, \beta_j)$, where α_i and β_j vary over the components of the support of α and β , where λ_i and λ_j are the respective weighting on them, and where $\iota(\alpha_i, \beta_j) \in \mathbb{N}$ is the usual geometric intersection number.

We write $d(\alpha, \beta) = \min_{i,j} \{d(\alpha_i, \beta_j)\}$, again where α_i and β_j vary over the components of α, β .

Given $\gamma \in V(\mathcal{G})$ we set $l(\gamma) = l_{\alpha\beta}(\gamma) = \max\{\iota(\alpha, \gamma), \iota(\beta, \gamma)\}$ (interpreting γ as a one-component multicurve of unit weight). One can think of $l(\gamma)$ as describing a “length” in a singular euclidean structure arising from α and β (cf. Section 5 of [B]).

Lemma 2.4. *Suppose that α, β are weighted multicurves with $\iota(\alpha, \beta) = 1$ and $d(\alpha, \beta) \geq 2$. Then there is some $\delta \in V(\mathcal{G})$ with $l(\delta) \leq R$ and such that $\iota(\gamma, \delta) \leq Rl(\gamma)$ for all $\gamma \in V(\mathcal{G})$ (where R is defined as above).*

Note that this is just a restating of Lemma 4.1 of [B] for this particular definition of R .

Proof. The proof is the same as that of Lemma 4.1 of [B]. Suppose first that $\alpha \cup \beta$ fills $\Sigma \setminus \Pi$. As in Section 5 of that paper, we construct a singular euclidean surface, tiled by rectangles, dual to $\alpha \cup \beta$. The cone angles are all multiples of π , and all cone singularities of angle π lie in Π . Thus, any trivial region, $\Delta \subseteq \Pi$, contains at most one cone point of angle less than 2π . Passing to a branched double cover over this cone point (if it exists) we are reduced to considering the case where all cone angles are at least 2π . But then the worst case is a round circle in the euclidean plane [Wei] which would give $\text{area}(\Delta) = \text{length}(\partial\Delta)^2/4\pi$. We can therefore set $h = 2(1/4\pi) = 1/2\pi$. Now apply Lemma 2.3, and

set δ to be a core curve of that annulus. The statement then follows exactly as in [B] (at the end of Section 5 thereof). (In [B], h was given inaccurately as $\pi/2$.)

If $\alpha \cup \beta$ does not fill $\Sigma \setminus \Pi$, we get instead a singular euclidean structure on a “smaller” surface, namely a region of Σ with each boundary component collapsed to a point. However, this process can only decrease ξ_1 and ξ_2 , so we again get an annulus of width at least η . (This case is the reason we needed a version of Lemma 2.3 when $3g + p = 4$. In the case where $(g, p) = (0, 4)$, note that $1/2$ is certainly greater than the required $\sqrt{2\pi}/120$.) \square

Given $r \geq 0$, set $L(\alpha, \beta, r) = \{\gamma \in V(\mathcal{G}) \mid l(\gamma) \leq r\}$. Note that the curve δ given by Lemma 2.4 lies in $L(\alpha, \beta, R)$.

Lemma 2.5. *Suppose that $2g + p \geq 195$. Suppose that α, β are weighted multicurves with $\iota(\alpha, \beta) = 1$ and $d(\alpha, \beta) \geq 2$. Then, the diameter of $L(\alpha, \beta, 2R)$ in \mathcal{G} is at most 20.*

Proof. Let δ be as given by Lemma 2.4. If $\gamma \in L(\alpha, \beta, 2R)$, then $l(\gamma) \leq 2R$, so $\iota(\gamma, \delta) \leq 2R^2$. If we knew that $16\iota(\gamma, \delta) \leq \xi_0^5$, then Corollary 2.2 with $q = 4$ would give $d(\gamma, \delta) \leq 10$ and the result would follow.

It is therefore sufficient that $16(2R^2) \leq \xi_0^5$. Recall that $R = (4/\sqrt{\pi})\xi_1\xi_2$, so this reduces to $32(4/\sqrt{\pi})^2\xi_1^2\xi_2^2 \leq \xi_0^5$, that is, $512\xi_1^2\xi_2^2 \leq \pi\xi_0^5$. In other words, we want

$$(*) \quad 512(2g + p - 1)^2(2g + p + 6)^2 \leq \pi(2g + p - 4)^5$$

which holds whenever $2g + p \geq 195$. \square

We now assume that $2g + p \geq 195$.

Recall that Lemma 4.3 of [B] states that $L(\alpha, \beta, R)$ has diameter bounded by some constant D (which there, depended on R). Since $L(\alpha, \beta, R) \subseteq L(\alpha, \beta, 2R)$, we have now verified Lemma 4.3 of [B] with $D = 20$. Recall that Lemma 4.2 of [B], more generally, placed a bound on the diameter of $L(\alpha, \beta, r)$ depending on r and R (specifically, $\text{diam } L(\alpha, \beta, r) \leq 2Rr + 2$). This was used in the proof of Lemma 4.12 of [B]. We can now use Lemma 2.5 above, in place of Lemma 4.2 of [B], to give a proof of Lemma 4.12 of [B] with the constant $4D$ now replaced by 40. We can now proceed as in [B] to prove Lemma 4.13 and Proposition 4.11 of that paper. In fact, the improvement on Lemma 4.12 allows us, respectively, to replace the constants $14D$ by $10D$ and $18D$ by $14D$, where $D = 20$. Thus, the original diameter bound of $18D$ of Proposition 4.11 of [B] now becomes 280.

Recall that Proposition 3.1 of [B] gives a criterion for hyperbolicity depending on a constant, K , in the hypotheses. The three clauses (1), (2) and (3) of those hypotheses were verified respectively by Lemma 4.10, Proposition 4.11 and Lemma 4.9. These respectively gave K bounded by $4D$, $18D$ and $2D$, which we can now replace by 80, 280 and 40. In particular, we have shown:

Proposition 2.6. *If $2g+p \geq 195$, then the curve graph $\mathcal{G}(g, p)$ satisfies the hypotheses of Proposition 3.1 of [B] with $K = 280$.*

For $2g + p \geq 195$, one can now explicitly estimate k from the proof of Proposition 3.1 of [B]. In fact, one can do better.

3. A CRITERION FOR HYPERBOLICITY

We give a self-contained account of a criterion for hyperbolicity which is related to, but simpler than, that used in [B]. In particular, it does not require the condition on moving centres (clause (2) of Proposition 3.1 of [B]) which complicated the argument there. Essentially the same statement can be found in Section 3.13 of [MS], though without a specific estimate for the hyperbolicity constant arising (or the final clause about Hausdorff distance). Our proof uses an idea to be found in [Gi], but bypasses use of the isoperimetric inequality. Since this criterion has many applications, this may be of some independent interest. For definiteness, we say that a space is k -hyperbolic if, in every geodesic triangle, each side lies in a k -neighbourhood of the union of the other two.

Proposition 3.1. *Given $h \geq 0$, there is some $k \geq 0$ with the following property. Suppose that G is a connected graph, and that for each $x, y \in V(G)$, we have associated a connected subgraph, $\mathcal{L}(x, y) \subseteq G$, with $x, y \in \mathcal{L}(x, y)$. Suppose that:*

(1) for all $x, y, z \in V(G)$,

$$\mathcal{L}(x, y) \subseteq N(\mathcal{L}(x, z) \cup \mathcal{L}(z, y), h),$$

and

(2) for any $x, y \in V(G)$ with $d(x, y) \leq 1$, the diameter of $\mathcal{L}(x, y)$ in G is at most h .

Then G is k -hyperbolic.

In fact, we can take any

$$k \geq (3m - 10h)/2,$$

where m is any positive real number satisfying

$$2h(6 + \log_2(m + 2)) \leq m.$$

Moreover, for all $x, y \in V(G)$, the Hausdorff distance between $\mathcal{L}(x, y)$ and any geodesic from x to y is bounded above by $m - 4h$.

Here, d is the combinatorial metric on G , and $N(., h)$ denotes h -neighbourhood. Note that we can assume that $\mathcal{L}(x, y) = \mathcal{L}(y, x)$ (on replacing $\mathcal{L}(x, y)$ with $\mathcal{L}(x, y) \cup \mathcal{L}(y, x)$). Note that the condition on m is monotonic: if it holds for m , it holds strictly for any $m' > m$.

Proof. Given any $x, y \in V(G)$, let $\mathcal{I}(x, y)$ be the set of all geodesics from x to y . Given any $n \in \mathbb{N}$, write

$$f(n) = \max\{d(w, \alpha) \mid (\exists x, y \in V(G)) d(x, y) \leq n, \alpha \in \mathcal{I}(x, y), w \in \mathcal{L}(x, y)\}.$$

In other words, $f(n)$ is the minimal $f \geq 0$ such that $\mathcal{L}(x, y) \subseteq N(\alpha, f)$ for any geodesic, α , connecting any two vertices x, y a distance at most n apart.

We first claim that $f(n) \leq (2 + \lceil \log_2 n \rceil)h$ (cf [Gi]). To see this, write $l = d(x, y) \leq n$ and $p = \lceil \log_2 l \rceil + 2$. Let $z \in V(G)$ be a “near midpoint” of α , that is, it cuts α into two subpaths, α^- and α^+ whose lengths differ by at most 1. By (1), $\mathcal{L}(x, y) \subseteq N(\mathcal{L}(x, z) \cup \mathcal{L}(z, y), h)$. We now choose near midpoints of each of the paths α^+ and α^- and then continue inductively. After at most $p - 1$ steps, we see that $\mathcal{L}(x, y) \subseteq N(\bigcup_{i=0}^{p-1} \mathcal{L}(x_i, x_{i+1}), (p - 1)h)$ where $x = x_0, x_1, \dots, x_l = y$ is the sequence of vertices along α . Applying (2) now gives $\mathcal{L}(x, y) \subseteq N(\alpha, ph)$, and so $f(n) \leq ph$ as claimed.

In fact, we aim to show that $f(n)$ is bounded purely in terms of h . We proceed as follows.

Let $t = f(n) + 2h + 1$. Choose any $w \in \mathcal{L}(x, y)$. Let $l_0 = \max\{0, d(w, x) - t\}$ and $l_1 = \max\{0, d(w, y) - t\}$. Since $l = d(x, y)$, we have $l \leq l_0 + l_1 + 2t$, and so we can find vertices x', y' in α cutting it into subpaths $\alpha = \alpha_0 \cup \delta \cup \alpha_1$, where $d(x, x') \leq l_0$, $d(x', y') \leq 2t$ and $d(y', y) \leq l_1$. If $x = x'$ we leave out α_0 , and/or if $y = y'$ we leave out α_1 . (We can always assume that $x' \neq y'$.)

Note that $d(w, \alpha_0) \geq d(w, x) - d(x, x') \geq d(w, x) - l_0$. Therefore, if $x \neq x'$, then $l_0 = d(w, x) - t$, and so $d(w, \alpha_0) \geq t$. But $d(x, x') \leq d(x, y) \leq n$ and so $\mathcal{L}(x, x') \subseteq N(\alpha_0, f(n))$. It follows that $d(w, \mathcal{L}(x, x')) \geq t - f(n) = 2h + 1$. In other words, if $x \neq x'$, then $d(w, \mathcal{L}(x, x')) \geq 2h + 1$. Similarly, if $y \neq y'$, then $d(w, \mathcal{L}(y', y)) \geq 2h + 1$. But

$$w \in \mathcal{L}(x, y) \subseteq N(\mathcal{L}(x, x') \cup \mathcal{L}(x', y') \cup \mathcal{L}(y', y), 2h)$$

and so $d(w, \mathcal{L}(x', y')) \leq 2h$. Now $d(x', y') \leq 2t$ and so $\mathcal{L}(x', y') \subseteq N(\delta, f(2t))$. Thus, $w \in N(\delta, f(2t) + 2h) \subseteq N(\alpha, f(2t) + 2h)$. Since w

was an arbitrary point of $\mathcal{L}(x, y)$, it follows that

$$f(n) \leq f(2t) + 2h = f(2f(n) + 4h + 2) + 2h,$$

Writing $F(n) = 2f(n) + 4h + 2$, we have shown that $F(n) \leq F(F(n)) + 4h$ for all n .

Now, from the earlier claim,

$$F(n) \leq 2((2 + \log_2 n)h) + 4h + 2 = 2h(4 + \log_2 n) + 2.$$

Suppose m is as in the statement of the theorem. Writing $r = m + 2$, we have $2h(6 + \log r) + 2 \leq r$, and so $F(n) + 4h \leq 2h(6 + \log_2 n) + 2 < n$ for any $n > r$.

In summary, we have shown that

$$F(n) \leq F(F(n)) + 4h$$

for all n , and that

$$F(n) + 4h < n$$

for all $n > r$. It follows that $F(n) \leq r$ for all n (otherwise, we have the contradiction $F(n) \leq F(F(n)) + 4h < F(n)$). It now follows that $f(n) \leq s$, where $s = \frac{r}{2} - 2h - 1 = \frac{m}{2} - 2h$.

We have shown that for all $x, y \in V(G)$ and $\alpha \in \mathcal{I}(x, y)$, we have $\mathcal{L}(x, y) \subseteq N(\alpha, s)$. It now follows also that $\alpha \subseteq N(\mathcal{L}(x, y), 2s)$. Since if $w \in \alpha$, then w cuts α into two subpaths, α^- and α^+ . Since $\mathcal{L}(x, y)$ is connected and contains x, y , we can find some $v \in \mathcal{L}(x, y)$ and $v^\pm \in \alpha^\pm$ with $d(v, v^\pm) \leq s$. Now $d(w, \{v^-, v^+\}) \leq s$, so $d(v, w) \leq 2s$. We deduce that $d(w, \mathcal{L}(x, y)) \leq 2s$ as required.

Now suppose that $x, y, z \in V(G)$ and that $\alpha \in \mathcal{I}(x, y)$, $\beta \in \mathcal{I}(x, z)$ and $\gamma \in \mathcal{I}(y, z)$. We have

$$\alpha \subseteq N(\mathcal{L}(x, y), 2s) \subseteq N(\mathcal{L}(x, z) \cup \mathcal{L}(z, y), 2s + h) \subseteq N(\beta \cup \gamma, k),$$

where

$$k = 3s + h \leq 3((r/2) - 2h - 1) + h = (3m - 10h)/2.$$

Thus, G is k -hyperbolic. \square

4. ESTIMATION OF CONSTANTS

Given Proposition 3.1 of this paper, we can extract information more efficiently from [B], and bypass much of the proof of Theorem 1.1. Given, $\alpha, \beta \in V(\mathcal{G}(g, p))$ with $d(\alpha, \beta) \geq 2$ and $t \in \mathbb{R}$, let $\Lambda_{\alpha\beta}(t) = L((e^t/\iota)\alpha, (e^{-t}/\iota)\beta, R)$, where $\iota = \iota(\alpha, \beta) > 0$.

Now, $\iota((e^t/\iota)\alpha, (e^{-t}/\iota)\beta) = 1$. Therefore if $2g + p \geq 195$, then by Lemma 2.4, $\Lambda_{\alpha\beta}(t) \neq \emptyset$. Let $\mathcal{L}(\alpha, \beta)(t)$ be the full subgraph of \mathcal{G} with vertex set $\Lambda_{\alpha\beta}(t)$. It is not hard to see that $\mathcal{L}(\alpha, \beta)(t)$ is connected.

(For example, the standard argument, going back to work of Lickorish, for showing that \mathcal{G} itself is connected effectively does this. This involves interpolating between two curves by a series of surgery operations, cf. Lemma 1.3 of [B] for example. These can only decrease the intersection number with any fixed curve.) It follows easily that $\mathcal{L}(\alpha, \beta) = \bigcup_{t \in \mathbb{R}} \mathcal{L}(\alpha, \beta)(t)$ is connected. Note that the vertex set of $\mathcal{L}(\alpha, \beta)$ is the “line” $\Lambda_{\alpha\beta} = \bigcup_{t \in \mathbb{R}} \Lambda_{\alpha\beta}(t)$ as defined in [B]. Note also that $\alpha, \beta \in \Lambda_{\alpha\beta}$. If $d(\alpha, \beta) \leq 1$, we set $\Lambda_{\alpha\beta} = \{\alpha, \beta\}$, so that $\mathcal{L}(\alpha, \beta)$ is a single vertex or edge.

We can now verify that the collection $(\mathcal{L}(\alpha, \beta))_{\alpha, \beta \in V(\mathcal{G})}$ satisfies the hypotheses of Proposition 3.1 here with $h = 40$. Condition (2) is immediate. For condition (1), let $\alpha, \beta, \gamma \in V(G)$. If these three curves all pairwise intersect, then we set $\tau = \frac{1}{2} \log_e(\iota(\alpha, \beta)\iota(\alpha, \gamma)/\iota(\beta, \gamma))$. As in Lemma 4.5 of [B], we see that if $t \leq \tau$, the diameter of $\mathcal{L}(\alpha, \beta)(t) \cup \mathcal{L}(\alpha, \gamma)(t)$ is at most 40 (since we can set $D = 20$). Similarly, if $t \geq \tau$ then $\mathcal{L}(\alpha, \beta)(t) \cup \mathcal{L}(\beta, \gamma)(t)$ has diameter at most 40. Thus, $\mathcal{L}(\alpha, \beta) \subseteq N(\mathcal{L}(\alpha, \gamma) \cup \mathcal{L}(\gamma, \beta), h)$ with $h = 40$. The cases where at least two of the curves α, β, γ are disjoint follow from a slight modification of this argument, as in [B]. This now gives $m \leq 1320$ and $k \leq 1780$. This shows that if $2g + p \geq 195$, then $\mathcal{G}(p, q)$ is 1780-hyperbolic.

In fact, since we are now only using Lemma 4.3 of [B], we can replace $2R$ by R in Lemma 2.5 here, so that the requirement $16(2R^2) \leq \xi_0^5$ becomes $16R^2 \leq \xi_0^5$, and so we can replace the resulting factor of 512 in (*) by 256. It is therefore sufficient that $2g + p \geq 107$. We have therefore shown that if $2g + p \geq 107$, then $\mathcal{G}(g, p)$ is 1780-hyperbolic.

We can deal with lower complexity surfaces using larger values of q from Corollary 2.2. In general, we require that $2^{q+4}(2g + p - 1)^2(2g + p + 6)^2 \leq \pi(2g + p - 4)^{q+1}$. For example, with $q = 5$, this is satisfied for $2g + p \geq 26$. This gives $D = 4(q + 1) = 24$, $h = 2D = 48$, $m \leq 1584$ and $k \leq 2136$. In other words, if $2g + p \geq 26$, then $\mathcal{G}(g, p)$ is 2064-hyperbolic. Similarly (with $q = 6$), if $2g + p \geq 14$, then $\mathcal{G}(g, p)$ is 2492-hyperbolic etc.

For the cases where $2g + p \leq 6$, we need to revert to previous arguments. The estimates and methods in [T] might give improvements for some of the lower complexities.

There is scope for other improvements in various directions. For the bounds on complexity for example, suppose $p = 0$. In the proof of Lemma 2.3 we don't have to worry about trivial regions, so we can easily obtain $l \leq 2\lambda$, allowing us to reset $\xi_2 = 2g + 2$. We can also reset $\xi_1 = 2g$. For Lemma 2.2, we could set $h = 1/4\pi$, further decreasing R by a factor of $\sqrt{2}$. In Lemma 1.3 of [B], we can eliminate the factor of

2 in the hypotheses, and thereby weaken those of Corollary 2.2 here to saying that $\iota(\alpha, \beta) \leq x_0^q$. The fact that we have replaced $2R$ by R , also gives us another factor of 2, so that our requirement, when $q = 4$, now becomes $R^2 \leq \xi_0^5$. Together these now give $8(2g)^2(2g+2)^2 \leq \pi(2g-4)^5$, that is, $4g^2(g+1)^2 \leq \pi(g-2)^5$, which holds for $g \geq 8$. In other words, $\mathcal{G}(g, 0)$ is 1780-hyperbolic for $g \geq 8$.

We remark that in [HePW], it is shown that every curve graph is “17-hyperbolic” in the sense that, for every geodesic triangle, there is a vertex a distance at most 17 from each of its sides. From this, one can easily derive a uniform hyperbolicity constant in the sense we have defined it.

REFERENCES

- [A] T.Aougab, *Uniform hyperbolicity of the graphs of curves* : preprint, 2012, posted at arXiv:1212.3160.
- [B] B.H.Bowditch, *Intersection numbers and the hyperbolicity of the curve complex* : J. reine angew. Math. **598** (2006) 105–129.
- [CRS] M.Clay, K.Rafi, S.Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences* : preprint, 2013, posted at arXiv:1302.5519.
- [Gi] R.H.Gilman, *On the definition of word hyperbolic groups* : Math. Z. **242** (2002) 529–541.
- [Gr] M.Gromov, *Hyperbolic groups* : in “Essays in group theory” Math. Sci. Res. Inst. Publ. No. 8, Springer (1987) 75–263.
- [Ha] W.Harvey, *Boundary structure of the modular group* : in “Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference” (ed. I.Kra, B.Maskit), Ann. of Math. Stud. No. 97, Princeton University Press (1981) 245–251.
- [HePW] S.Hensel, P.Przytycki, R.C.H.Webb, *Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs* : preprint, 2013, posted at arXiv:1301.5577.
- [MM1] H.A.Masur, Y.N.Minsky, *Geometry of the complex of curves I: hyperbolicity* : Invent. Math. **138** (1999), 103–149.
- [MM2] H.A.Masur, Y.N.Minsky, *Geometry of the complex of curves II: hierarchical structure* : Geom. Funct. Anal. **10** (2000) 902–974.
- [MS] H.Masur, S.Schleimer, *The geometry of the disk complex* : J. Amer. Math. Soc. **26** (2013) 1–62.
- [T] R.Tang, *Covering maps and hulls in the curve complex* : Thesis, Warwick, 2013.
- [Web] R.C.H.Webb, *A short proof of the bounded geodesic image theorem* : preprint, 2013, posted at arXiv:1301.6187.
- [Wei] A.Weil, *Sur les surfaces à courbure négative* : C. R. Acad. Sci. Paris **182** (1926) 1069–1071.

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